2.2 Inverse of a Matrix

Steps to solve $3x = 5 \rightarrow x = \frac{5}{3}$?

2.2 Inverse of a Matrix

Steps to solve $3x = 5 \to x = \frac{5}{3}$?

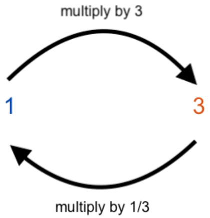
$\frac{1}{3}(3x) = \frac{1}{3}(5)$

$(\frac{1}{3}3)x = \frac{5}{3}$

$(1)x = \frac{5}{3}$

$x = \frac{5}{3}$

Action view:



Undo view: $3 \cdot \frac{1}{3} = 1$ and $\frac{1}{3} \cdot 3 = 1$

2.2 Algebra: Inverse of a Matrix

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Steps to solve $A\vec{x} = \vec{b}$

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if A has an inverse A^{-1} that will "undo" A

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Multiply both sides: $A^{-1}(A\vec{x}) = A^{-1}\vec{b}$

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Reorder by associativity: $(A^{-1}A)\vec{x} = A^{-1}\vec{b}$

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Cancel A by its inverse: $I_{n\times n}\vec{x} = A^{-1}\vec{b}$

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If A^{-1} exists then $A\vec{x} = \vec{b}$ has this unique solution for each $\vec{b}$ and $[A|\vec{b}] \to [I_{n\times n}|A^{-1}\vec{b}]$

Multiplication of Matrices in 2.1 (Extension of 1.4)

Columns of B Method: $AB = \begin{bmatrix} A.\text{col}1B & \dots & A.\text{col}nB \end{bmatrix}$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 6 & 7\\8 & 9 & 10\end{bmatrix} = \left[\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\8\end{bmatrix} \quad \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}6\\9\end{bmatrix} \quad \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}7\\10\end{bmatrix}\right]$$

$$= \begin{bmatrix}1\cdot 5 + 2\cdot 8 & 1\cdot 6 + 2\cdot 9 & 1\cdot 7 + 2\cdot 10\\3\cdot 5 + 4\cdot 8 & 3\cdot 6 + 4\cdot 9 & 3\cdot 7 + 4\cdot 10\end{bmatrix}$$

Dot Product Method: $AB_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$ = [row$i$ A] · [column$j$ B]

$$= \begin{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix}\cdot\begin{bmatrix}5\\8\end{bmatrix} & \begin{bmatrix}1 & 2\end{bmatrix}\cdot\begin{bmatrix}6\\9\end{bmatrix} & \begin{bmatrix}1 & 2\end{bmatrix}\cdot\begin{bmatrix}7\\10\end{bmatrix}\\ \begin{bmatrix}3 & 4\end{bmatrix}\cdot\begin{bmatrix}5\\8\end{bmatrix} & \begin{bmatrix}3 & 4\end{bmatrix}\cdot\begin{bmatrix}6\\9\end{bmatrix} & \begin{bmatrix}3 & 4\end{bmatrix}\cdot\begin{bmatrix}7\\10\end{bmatrix}\end{bmatrix}$$

Inverse of a 2×2 Matrix

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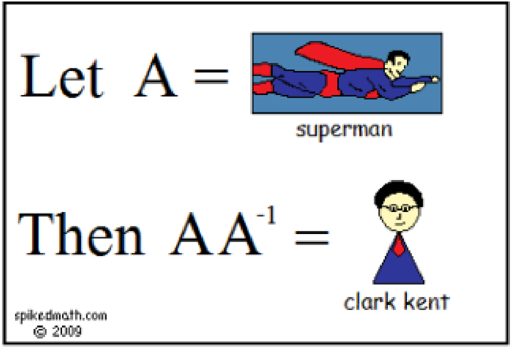
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ exists when $ad - bc \neq 0$, called the determinant of A_{2x2} or $|A|$

Use linear algebra to find the identity of superman.



http://spikedmath.com/042.html

Solving a System

$$\begin{aligned} 2x + y &= 5 \\ 3x + y &= 6 \end{aligned}$$

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computer speed

Finding the Inverse

If A is invertible then $A \to I_{n\times n}$ and the same row operations that turn A to $I_{n\times n}$ turn $I_{n\times n}$ to A^{-1}.

$$[A|I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{r_2' = -\frac{c}{a} r_1 + r_2} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & -\frac{bc}{a} + d & -\frac{c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{r_2' = \frac{a}{ad-bc} r_2} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{r_1' = -b r_2 + r_1} \left[\begin{array}{cc|cc} a & 0 & 1 + \frac{bc}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{r_1' = \frac{1}{a} r_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

so $A^{-1} = \frac{1}{ad-bc} \left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]$

Sherlock Holmes: *Elementary my dear Watson*

An *elementary matrix* is one where we perform a single elementary row operation on an identity matrix.

Finding the Inverse Via Elementary Row Operations

If A is invertible then $A \to I_{n\times n}$ and the same row operations that turn A to $I_{n\times n}$ turn $I_{n\times n}$ to A^{-1}

Row operations can be written as matrix multiplications:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

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Keep track of row operations that turn A to $I_{n\times n}$

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Keep track of row operations that turn A to $I_{n\times n}$

$E_p \ldots E_2 E_1 A = I$ so $A^{-1} = E_p \ldots E_2 E_1$

Use Elementary Row Operation to Find Inverse

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ scales row 2 by 3. To undo scale row 2 by $\frac{1}{3}$.

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$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ swaps row 2 and row 3. To undo swap again.

$$B^{-1} = B = E_{r_2 \leftrightarrow r_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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Invertibility of Numbers and Functions

Every number but ___ is invertible

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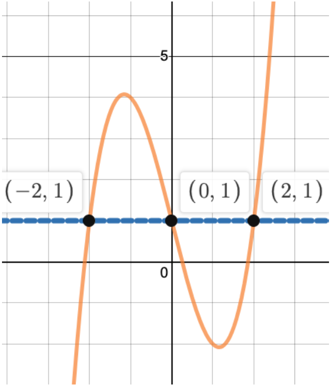
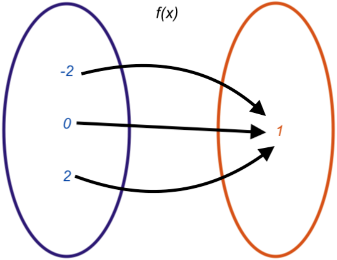
Every number but ___ is invertible

$$f(x) = x^3 - 4x + 1$$





Invertibility of Square Matrices

Consider an $n \times n$ matrix A:

A is invertible $\leftrightarrow$ $A\vec{x} = \vec{b}$ has a *unique* solution for each $\vec{b} \in \mathbb{R}^n$ $\leftrightarrow$ A has n pivots

Invertibility of Square Matrices: Column by Column

Find $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}^{-1}$

We want to solve $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} p \\ s \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} q \\ t \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} r \\ u \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{ReducedRowEchelonForm} \left[\begin{array}{ccc|c} 1 & 0 & 0 & p \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & v \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{ReducedRowEchelonForm} \left[\begin{array}{ccc|c} 1 & 0 & 0 & r \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & x \end{array}\right]$$

Invertibility of Square Matrices: Column by Column

Find $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}^{-1}$

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$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} p \\ s \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} q \\ t \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} r \\ u \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{ReducedRowEchelonForm} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & p & q & r \\ 0 & 1 & 0 & s & t & u \\ 0 & 0 & 1 & v & w & x \end{array}\right]$$

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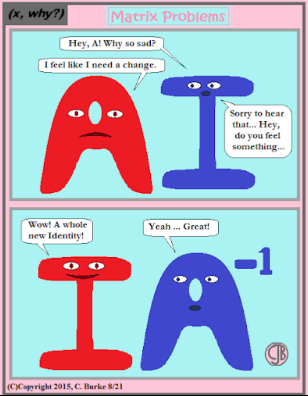
$$\underbrace{\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]}_{[A|I_{n\times n}]} \xrightarrow{ReducedRowEchelonForm} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]}_{[I_{n\times n}|A^{-1}]}$$

Invertibility Terminology for Square Matrices

Nonsingular: is invertible (row equivalent to I_{nxn})

Singular matrix: not invertible

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is singular or not invertible

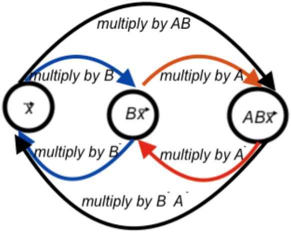


Socks-Shoes Property for $(AB)^{-1}$

A and B invertible

$$(AB)^{-1} = B^{-1}A^{-1}$$

AB put on socks B and then shoes A. When we apply the inverse, we take shoes A^{-1} off and then socks B^{-1}



Why are the columns of an invertible matrix l.i.?

Without presuming Theorem 8 in 2.3

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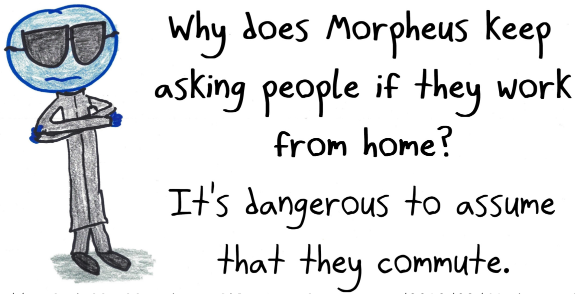
problematic reasoning: If the 2 columns of A are multiples the determinant will be 0

incomplete reasoning: the columns of A are l.i. because $A\vec{x} = \vec{0}$ has only the trivial solution when A is invertible (why?).



https://mathwithbaddrawings.files.wordpress.com/2018/02/41.jpg?w=2200