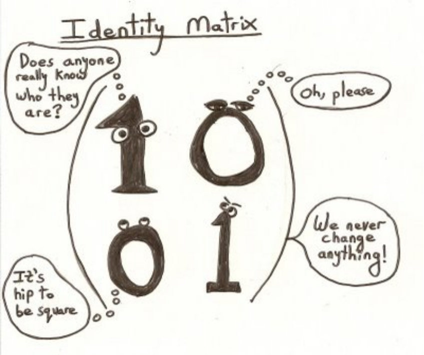
2.3 Characterizations of Invertible Matrices

Assume $A_{n\times n}$ (square) is invertible. What can we say about:

- solutions of systems of equations involving A as the coefficient matrix ($A\vec{x} = \vec{0}, A\vec{x} = \vec{b}$)?
- columns pivots of A? row pivots of A?
- Gauss-Jordan reduction of A (what A is *row equivalent* to)?



Credit: disconsolations. Retrieved from

http://sciencejokes.tumblr.com/post/15121642099/httpsciencejokestumblrcom

Why? Why Not?

theoretical multiplication arguments:

$$AB = \begin{bmatrix} A.\text{col}1B & \dots & A.\text{col}nB \end{bmatrix}$$

OR

Multiply both sides on side it makes sense: $A^{-1}(A\vec{x}) = A^{-1}\vec{b}$
Reorder parentheses by associativity: $(A^{-1}A)\vec{x} = A^{-1}\vec{b}$
Cancel A by its inverse: $I_{n\times n}\vec{x} = A^{-1}\vec{b}$
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- A not square must be missing a row pivot or a column pivot (but not necessarily both)

Invertible Matrix Theorem for $A_{n\times n}$

The following are equivalent (TFAE):

- A is an invertible matrix
- A is row equivalent to the $n \times n$ identity matrix
- A has n pivot positions
- $A\vec{x} = \vec{0}$ has only the trivial solution
- columns of A form a linearly independent set
- $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$
- columns of A span $\mathbb{R}^n$
- there is an $n \times n$ matrix C such that $CA = I$
- there is an $n \times n$ matrix D such that $AD = I$
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$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \quad \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & 0\\0 & 0\\0 & 0\end{bmatrix}$$

A B C D E F

Given $A_{n\times n}$ (square), can $A\vec{x} = \vec{0}$ have only the trivial solution?

a) no that statement is impossible
b) yes the columns of A are l.i. but we can't say anything more
c) yes the columns of A are l.i. and A has n pivot rows
d) yes the columns of A are l.i. and A has n pivot columns
e) both c) and d)

http://www.mathfunny.com/images/mathpics-mathjoke-haha-humor-pun-mathmeme-meme-joke-math-problems-harry-variable-equation-xyz.jpg

If A is an invertible $n \times n$ matrix, and $\vec{x}$ and $\vec{b}$ are $1 \times n$ vectors, then the matrix-vector equation $A\vec{x} = \vec{b}$ has a unique solution

a) True
b) False

Silliness: Who writes inverse?

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b) False

Silliness: Who writes inverse?

A backwards poet.

What Makes You Invertible

In verse

What Makes You Invertible

In verse

Music by One Direction & idea adapted from Art Benjamin

Interpreted by Dr. Sarah and Joel Landsberg

Baby you'll light up if one of these facts is so,
but you'll need n square columns and rows:

- Like when $\mathbb{R}^n$ is the span of the matrix columns
- That's when you know oh-oh invertible!
- If always you uniquely solve $A\vec{x}$ is $\vec{b}$
- Or if your columns have no linear dependency
- Or if matrix reduces to identity

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Not zero - no no
That makes it not invertible!

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Shout out if one of these facts is so...
but you'll need n square columns and rows:

- Like when your matrix determinant's non-zero

Is when you know oh-oh—that makes it invertible!

Condition Number to Characterize A^{-1}

- computer algebra software programs like Maple will output a *condition number* corresponding to a matrix
- The order k in scientific notation (10^k) is useful, rather than the number, which can be different in different programs

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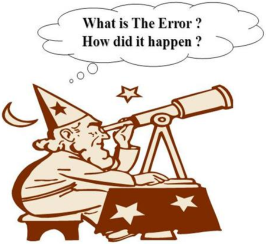
- The order k in scientific notation (10^k) is useful, rather than the number, which can be different in different programs
- measures asymptotically worst case scenario that we may lose up to k digits in roundoff errors
- issue with decimals in the matrix, not with Maple
- using r digits gets at least $r - k$ accuracy [ex: 21-19=2]



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If the condition number of a square matrix with fractional entries is 3.5×10^6 then...

a) we should use 8 decimal places in our measurements of $\vec{b}$ if we want solutions to $A\vec{x} = \vec{b}$ to be accurate to 2 decimal places

b) the matrix is invertible

c) both of the above

d) none of the above