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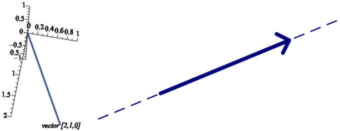
geometry 

https://www.geogebra.org/m/yscdejye

Homogeneous Equations $A\vec{x} = \vec{0}$

- The *trivial solution* is the $\vec{x} = \vec{0}$ solution to $A\vec{x} = \vec{0}$
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- Solutions are the span of one or more vectors, like
  - $t\vec{v}_1$ infinite line through the origin
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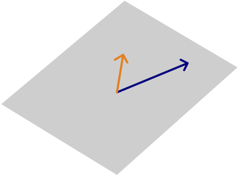
    infinite plane through the origin
  - number of free variables in solutions determines the geometry, e.g. 3 free variables gives all of $\mathbb{R}^3$ or a higher dimensional infinite volume (subspace)



Nonhomogeneous Equations

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$$\vec{x} = \begin{bmatrix} 1+2t \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = t\vec{v_1} + \vec{v_2}$$

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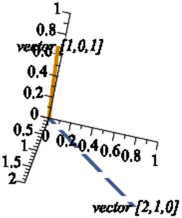

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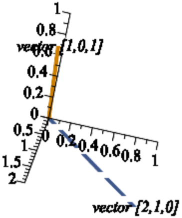

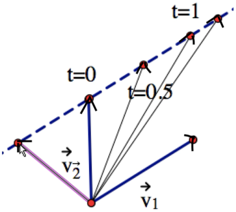

Separate and Factor

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- Solve homogeneous equation $A\vec{x} = \vec{0}$ to find the intersections of the rows. If they intersect in more than the trivial solution, then we can express the linear space using the free variables:
  - Gaussian elimination to row echelon form
  - set any free variables (variables without pivots) as parameters
  - use the rows to solve for any variables with pivots in terms of the parameters
  - write out the solutions in vector form $\vec{x}$
  - factor out the free variables to see the solutions as the span of one or more vectors
- For $A\vec{x} = \vec{b}$, where $\vec{b} \neq 0$, if it is consistent with infinite solutions, then we can express solutions as $\vec{x} = \vec{x}_h + \vec{x}_p$.

  For example: $t\vec{v}_1 + \vec{v}_2 = t \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \end{bmatrix} + \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \end{bmatrix}$

Evelyn Boyd Granville, h Coefficients

 

Image 1 Credit: `http://www.visionaryproject.org/granvilleevelyn/`
Image 2 Credit: Marge Murray. Courtesy of Evelyn Boyd Granville

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If $-h^2+1$ is nonzero then $\vec{x}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, only the trivial solution.

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If $-h^2 + 1$ is nonzero then $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, only the trivial solution.

If $-h^2 + 1 = 0$ then $h = \pm 1$ and ∞ solutions.

When $h = 1$ we have line $\vec{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

When $h = -1$ we have line $\vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Algebra and Geometry of Parameterized Solutions

Suppose that the augmented matrix for a system reduces to $\begin{bmatrix} 1 & -4 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Describe the solutions, the intersections of the rows, geometrically and in parametric vector form.

a) a line $t \begin{bmatrix} 1 \\ -4 \\ 5 \\ 6 \end{bmatrix}$ in $\mathbb{R}^4$

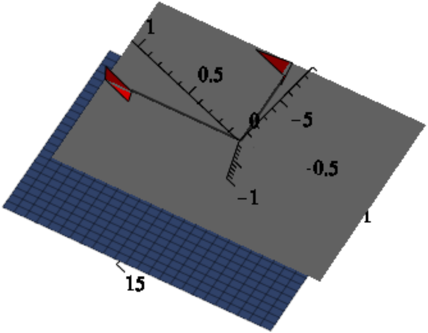
b) a plane $s \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$

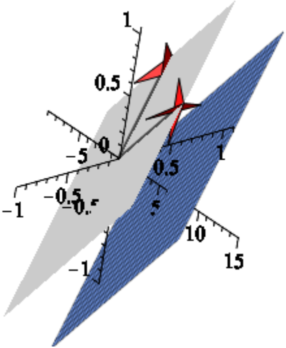
c) another line

d) another plane

e) other

solutions of $\begin{bmatrix} 1 & -4 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$: plane $s\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$





Number of Solutions

A homogeneous linear-system has how many solutions?

a) 0, 1 or infinite
b) 0 or infinite
c) 1 or infinite
d) other

OED: *Origin Early 17th century (as homogeneity): from medieval Latin homogeneus, from Greek homogenes, from homos 'same' + genos 'kind'.*

Geometry of Solutions

For two column vectors $\vec{v}_1$ and $\vec{v}_2$, $\{c_1\vec{v}_1 + \vec{v}_2$ with c_1 real$\}$ is

a) a collection of vectors whose tips lie on the line parallel to $\vec{v}_1$ and through the tip of $\vec{v}_2$

b) a collection of vectors whose tips lie on the line parallel to $\vec{v}_2$ and through the tip of $\vec{v}_1$

c) a line because c_1 is free, but we can't say any more about it

d) more than one of the above

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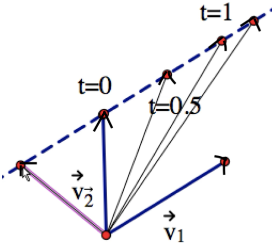

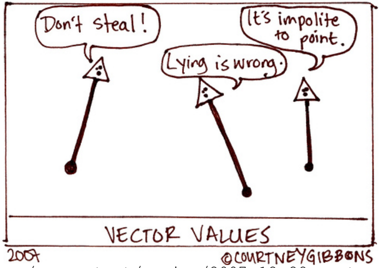

brownsharpie.courtneygibbons.org/wp-content/comics/2007-12-28-vector-values-1.jpg

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{bmatrix} \xrightarrow[r_3'=-7r_1+r_3]{r_2'=-4r_1+r_2}$$

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consistent—yes! x_3 has no pivot so parameter t. Backsubtitute variables attached to pivots in terms of parameters:

row 2: $-3x_2 - 6x_3 = 0$ so $-3x_2 - 6\,t = 0$, i.e. $x_2 = -2t$

row 1: $x_1 + 2x_2 + 3x_3 = 0$ so $x_1 + 2(-2t) + 3(t) = 0$, i.e. $x_1 = -2(-2t) - 3t = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$