1.2 Gaussian, Row Reduction and Echelon Forms

$$\begin{aligned} x + 2y + z &= 3 \\ 3x + 6y - z &= 4 \\ 5x + 10y + z &= 10 \end{aligned} \quad \leftrightarrow \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix}$$

Elementary Row Operations

replacement interchange scaling

Leading Entries, Pivots and Pivot Columns

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix} \xrightarrow[]{\substack{r_2' = -3r_1 + r_2 \\ r_3' = -5r_1 + r_3}} \begin{bmatrix} \boxed{1} & \boxed{2} & 1 & 3 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

We used this *leading entry* to create zeros below it. The positions of such are called *pivot positions* and the columns in the original matrix that correspond are called *pivot columns*.

Leading Entries, Pivots and Pivot Columns

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The 2 here is not a pivot.

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix} \xrightarrow[r_3' = -5r_1 + r_3]{r_2' = -3r_1 + r_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\xrightarrow{r_3' = -r_2 + r_3} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

-4 is another pivot. The 1st and 3rd columns in the original matrix are pivot columns

Free Parameters

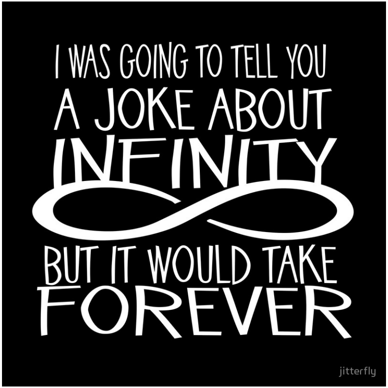
$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x + 2y + z &= 3 \\ z &= \frac{5}{4} \end{aligned}$$

no pivot in 2nd column, so y is a *free parameter*—parameterize

$$\begin{aligned} x + 2y + z &= 3 \\ y &= t \\ z &= \frac{5}{4} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{7}{4} - 2t \\ t \\ \frac{5}{4} \end{pmatrix}$$



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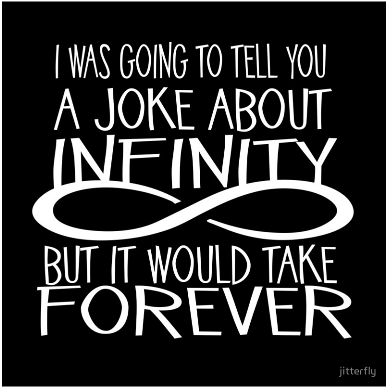

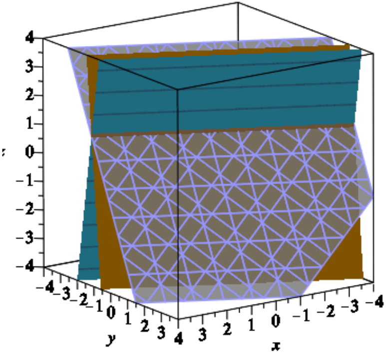

Image Credit: jitterfly

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Row Echelon Form

A matrix is in *row echelon form* (ref) if

1. all nonzero rows are above any rows of all zeros
2. each leading entry of a row is in a column to the right of the leading entry of the row above it
3. all entries in a column below a leading entry are zeros

Maple: GaussianElimination(A)

ref solutions

Which augmented matrix is not in ref/Gaussian form?

a) $$\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & -5 & 7 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

b) $$\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & 0 & 7 \\ 0 & 0 & 4 & 0 & 14 \end{bmatrix}$$

c) $$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ref solutions

Which augmented matrix is not in ref/Gaussian form?

a) $$\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & -5 & 7 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

c) $$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) $$\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & 0 & 7 \\ 0 & 0 & 4 & 0 & 14 \end{bmatrix}$$

consider how many solutions, if any, and consider how many free variables, if any

a) $\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & -5 & 7 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

no solution, no concurrent intersections in $\mathbb{R}^4$

c) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

unique solution, a point in $\mathbb{R}^3$

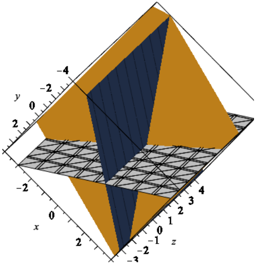
b) $\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

infinitely many solutions, a plane in $\mathbb{R}^4$

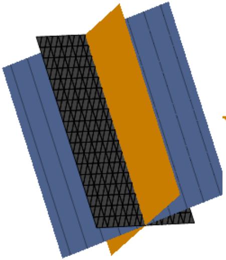
underdetermined versus *overdetermined*

Solutions and Pivots

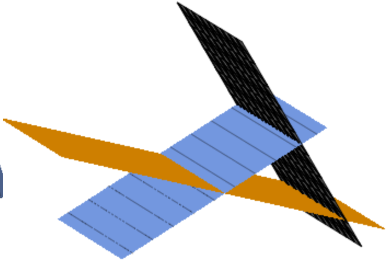
3 equations and 3 unknowns. How many pivot positions and pivot columns does each augmented matrix have?



1 solution
corner of room



infinite solutions
book spine



0 solutions
hands + table

Solutions and Pivots

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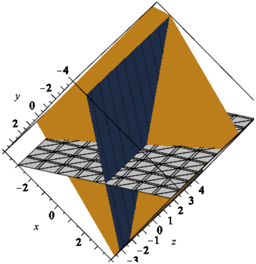
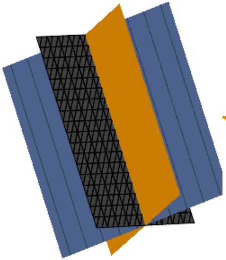


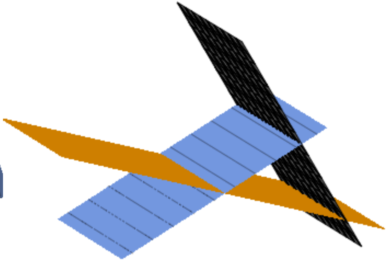
1 solution
corner of room

3 pivots



infinite solutions
book spine

2 pivots



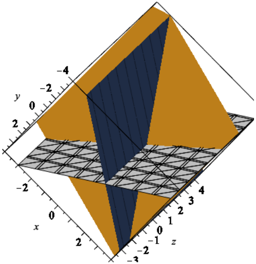
0 solutions
hands + table

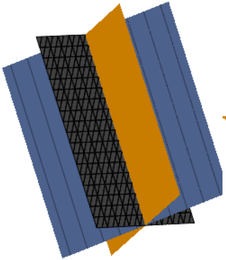
3 pivots, one in =

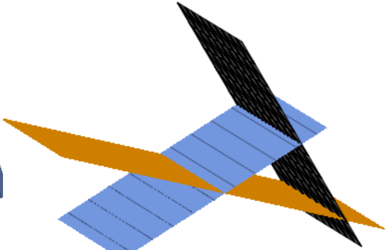
any linear system has 0, 1, ∞ solutions

Solutions and Pivots

3 equations and 3 unknowns. How many pivot positions and pivot columns does each augmented matrix have?







1 solution corner of room	infinite solutions book spine	0 solutions hands + table
3 pivots	2 pivots	3 pivots, one in =

any linear system has 0, 1, ∞ solutions : think of pivots!
geometry!
can an underdetermined system ever have 1 solution?

Reduced Row Echelon Form

A matrix is in *reduced row echelon form* (rref) if

1. all nonzero rows are above any rows of all zeros
2. each leading entry of a row is in a column to the right of the leading entry of the row above it
3. all entries in a column below a leading entry are zeros.
4. **the leading entry in each nonzero row is a 1**
5. **each leading 1 is the only nonzero entry in its column.**

Gauss-Jordan elimination
Maple: ReducedRowEchelonForm(A)

Moving from Gaussian to Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving from Gaussian to Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2' = \frac{1}{4} r_2} \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving from Gaussian to Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2' = \frac{1}{4} r_2} \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_1' = -2r_2 + r_1} \begin{bmatrix} 1 & 0 & -6 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving from Gaussian to Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_3' = \frac{1}{2} r_3} \begin{bmatrix} 1 & 0 & -6 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[r_1' = 6r_3 + r_1]{r_2' = -2r_3 + r_2} \begin{array}{cccc} x & y & z & = \end{array} \begin{bmatrix} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Flops

A flop, for floating point operation, is a measure of $+, -, \times, \div$

- For an $n \times (n+1)$ matrix, it can take as many as $\frac{2n^3}{3} + \frac{n^2}{2} - \frac{7n}{6}$ flops to apply Gaussian elimination to reach the row echelon form (the *forward phase*).
- The *backwards phase* from row echelon form to reduced row echelon form (making use of all the zeros) can take another n^2 flops.

3×4 from ref to rref?

Describe the Solution

Describe the solution set to the *homogeneous* linear system with ReducedRowEchelonForm(A)

$$\left[\begin{array}{c|c} & 0 \\ A & 0 \\ & 0 \end{array}\right] \xrightarrow{\textit{RREF operations}} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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We have two free variables and so we know we have infinitely many solutions. Geometrically, a plane in $\mathbb{R}^4$. Parameterically, $x_3 = s$ and $x_4 = t$.

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$$\begin{aligned} x_1 + 0x_2 - 3x_3 - x_4 &= 0 \\ 0x_1 + x_2 - x_3 + 0x_4 &= 0 \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

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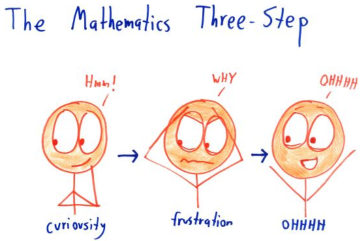
$$\begin{aligned} x_1 + 0x_2 - 3x_3 - x_4 &= 0 \\ 0x_1 + x_2 - x_3 + 0x_4 &= 0 \\ x_3 &= s \\ x_4 &= t \end{aligned} \qquad \begin{aligned} x_1 &= 3s + t \\ x_2 &= s \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

Use Gaussian on the following augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 3 & 3 \\ 2 & 2 & h & 4 \end{bmatrix}?$$

a) it takes at least 3 elementary row operations to get to Gaussian here
b) from Gaussian we can see that we have full pivots for all h
c) from Gaussian we can see that some h give us no solutions and a missing pivot
d) more than one of the above is true
e) none of the above



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Making CONNECTIONS

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