Evelyn Boyd Granville
second Black woman we know of—PhD in mathematics
Image 1 Credit: `http://www.visionaryproject.org/granvilleevelyn/`
Image 2 Credit: Marge Murray. Courtesy of Evelyn Boyd Granville

*...this was the most interesting job of my lifetime—to be a member of a group responsible for writing computer programs to track the paths of vehicles in space*

Evelyn Boyd Granville

second Black woman we know of—PhD in mathematics

Image 1 Credit: http://www.visionaryproject.org/granvilleevelyn/

Image 2 Credit: Marge Murray. Courtesy of Evelyn Boyd Granville

*...this was the most interesting job of my lifetime—to be a member of a group responsible for writing computer programs to track the paths of vehicles in space*

*Rabbits and chickens have been placed in a cage. You count 48 feet and seventeen heads*. Let $x$ = rabbits, $y$= chickens. heads?


Evelyn Boyd Granville
second Black woman we know of—PhD in mathematics
Image 1 Credit: http://www.visionaryproject.org/granvilleevelyn/
Image 2 Credit: Marge Murray. Courtesy of Evelyn Boyd Granville

*...this was the most interesting job of my lifetime—to be a member of a group responsible for writing computer programs to track the paths of vehicles in space*

*Rabbits and chickens have been placed in a cage. You count 48 feet and seventeen heads*. Let $x$ = rabbits, $y$= chickens. heads? feet?

Evelyn Boyd Granville
second Black woman we know of—PhD in mathematics

Image 1 Credit: `http://www.visionaryproject.org/granvilleevelyn/`
Image 2 Credit: Marge Murray. Courtesy of Evelyn Boyd Granville

*...this was the most interesting job of my lifetime—to be a member of a group responsible for writing computer programs to track the paths of vehicles in space*

*Rabbits and chickens have been placed in a cage. You count 48 feet and seventeen heads*. Let $x$ = rabbits, $y$= chickens. heads? feet? solve—three different methods?

**Elementary Row Operations**

1. (Replacement) Replace one row by the sum of itself and a multiple of another row [like $r_2' = -3r_1 + r_2$].
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

**Gaussian Elimination—Roughly**

1. Use $x_1$ in the first equation to eliminate the other $x_1$ terms below it
2. Ignore eq 1 and use $x_2$ in the 2nd equation to eliminate the terms below it
3. ... triangular (interchange as needed)

## *EBG Using Gaussian Elimination (Echelon Form)*

$$\begin{aligned} x + y &= 17 \\ 4x + 2y &= 48 \end{aligned} \qquad \begin{matrix} x & y & = \\ \end{matrix} \begin{bmatrix} 1 & 1 & 17 \\ 4 & 2 & 48 \end{bmatrix}$$

(Replacement) Replace a row by the sum of itself and a multiple of another row:

# *EBG Using Gaussian Elimination (Echelon Form)*

$$\begin{aligned} x + y &= 17 \\ 4x + 2y &= 48 \end{aligned} \qquad \begin{matrix} x & y & = \\ \end{matrix} \begin{bmatrix} 1 & 1 & 17 \\ 4 & 2 & 48 \end{bmatrix}$$

(Replacement) Replace a row by the sum of itself and a multiple of another row: adding -4·row 1 to row 2 ($r_2' = -4r_1 + r_2$)

## EBG Using Gaussian Elimination (Echelon Form)

$$\begin{aligned} x + y &= 17 \\ 4x + 2y &= 48 \end{aligned} \qquad \begin{matrix} x & y & = \\ \end{matrix} \begin{bmatrix} 1 & 1 & 17 \\ 4 & 2 & 48 \end{bmatrix}$$

(Replacement) Replace a row by the sum of itself and a multiple of another row: adding -4·row 1 to row 2 ($r_2' = -4r_1 + r_2$)

$$\begin{bmatrix} x & y & 17 \\ -4 \cdot x + 4x & -4 \cdot y + 2y & -4 \cdot 17 + 48 \end{bmatrix} = \begin{bmatrix} x & y & 17 \\ 0 & -2y & -20 \end{bmatrix}$$

## *EBG Using Gaussian Elimination (Echelon Form)*

$$x + y = 17$$
$$4x + 2y = 48$$

$$\begin{array}{ccc} x & y & = \end{array}$$
$$\begin{bmatrix} 1 & 1 & 17 \\ 4 & 2 & 48 \end{bmatrix}$$

(Replacement) Replace a row by the sum of itself and a multiple of another row: adding -4·row 1 to row 2 ($r_2' = -4r_1 + r_2$)

$$\begin{bmatrix} x & y & 17 \\ -4 \cdot x + 4x & -4 \cdot y + 2y & -4 \cdot 17 + 48 \end{bmatrix} = \begin{bmatrix} x & y & 17 \\ 0 & -2y & -20 \end{bmatrix}$$

- ~~(0x+0y=nonzero) **inconsistent**, 0 concurrent solutions~~

# *EBG Using Gaussian Elimination (Echelon Form)*

$$\begin{aligned} x + y &= 17 \\ 4x + 2y &= 48 \end{aligned} \qquad \begin{matrix} x & y & = \end{matrix} \\ \begin{bmatrix} 1 & 1 & 17 \\ 4 & 2 & 48 \end{bmatrix}$$

(Replacement) Replace a row by the sum of itself and a multiple of another row: adding -4·row 1 to row 2 ($r_2' = -4r_1 + r_2$)

$$\begin{bmatrix} x & y & 17 \\ -4 \cdot x + 4x & -4 \cdot y + 2y & -4 \cdot 17 + 48 \end{bmatrix} = \begin{bmatrix} x & y & 17 \\ 0 & -2y & -20 \end{bmatrix}$$

- ~~(0x+0y=nonzero) **inconsistent**, 0 concurrent solutions~~
- row 2: -2y=-20 so y=10

# *EBG Using Gaussian Elimination (Echelon Form)*

$$x + y = 17 \qquad \begin{matrix} x & y & = \end{matrix}$$
$$4x + 2y = 48 \qquad \begin{bmatrix} 1 & 1 & 17 \\ 4 & 2 & 48 \end{bmatrix}$$

(Replacement) Replace a row by the sum of itself and a multiple of another row: adding -4·row 1 to row 2 ($r_2' = -4r_1 + r_2$)

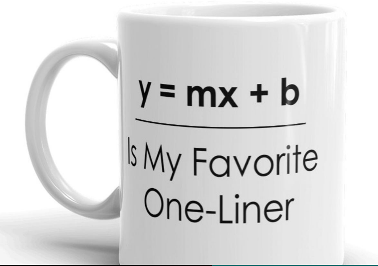
$$\begin{bmatrix} x & y & 17 \\ -4 \cdot x + 4x & -4 \cdot y + 2y & -4 \cdot 17 + 48 \end{bmatrix} = \begin{bmatrix} x & y & 17 \\ 0 & -2y & -20 \end{bmatrix}$$

- ~~(0x+0y=nonzero) **inconsistent**, 0 concurrent solutions~~
- row 2: -2y=-20 so y=10
  row 1: x+y=17, so x=17-y=17-10=7
  (7, 10) is the *unique* solution (point in $\mathbb{R}^2$)

# *Evelyn Boyd Granville in Maple & with Children*

```
with(LinearAlgebra): with(plots):
implicitplot(x+y=17, 4*x+2*y=48,x=-10..10, y = 0..40);
EBG:=Matrix([[1,1,17],[4,2,48]]);
GaussianElimination(EBG);
ReducedRowEchelonForm(EBG);
```

Children?



# *h as Coefficients of Variables*

$x + hy = 0$
$hx + y = 0$
Use Gaussian elimination $-h$ equation 1 + equation 2
We'll use an augmented matrix instead of writing the full equations each time.

# $h$ as Coefficients of Variables

$x + hy = 0$
$hx + y = 0$
Use Gaussian elimination $-h$ equation 1 + equation 2
We'll use an augmented matrix instead of writing the full equations each time.

Use $x$ term in eq1 to eliminate terms below it via $r_h' = cr_1 + r_k$

$$\begin{bmatrix} 1 & h & 0 \\ h & 1 & 0 \end{bmatrix}$$

## *h* as Coefficients of Variables

$x + hy = 0$
$hx + y = 0$
Use Gaussian elimination $-h$ equation 1 + equation 2
We'll use an augmented matrix instead of writing the full equations each time.

Use $x$ term in eq1 to eliminate terms below it via $r'_h = cr_1 + r_k$

$$\begin{bmatrix} 1 & h & 0 \\ h & 1 & 0 \end{bmatrix} \xrightarrow{r'_2 = -hr_1 + r_2} \begin{bmatrix} 1 & h & 0 \\ ? & ? & ? \end{bmatrix} =$$

# $h$ as Coefficients of Variables

$x + hy = 0$
$hx + y = 0$
Use Gaussian elimination $-h$ equation 1 + equation 2
We'll use an augmented matrix instead of writing the full equations each time.

Use $x$ term in eq1 to eliminate terms below it via $r'_h = cr_1 + r_k$

$$\left[\begin{array}{ccc} 1 & h & 0 \\ h & 1 & 0 \end{array}\right] \xrightarrow{r'_2 = -hr_1 + r_2} \left[\begin{array}{ccc} 1 & h & 0 \\ ? & ? & ? \end{array}\right] = \underbrace{\left[\begin{array}{ccc} 1 & h & 0 \\ 0 & -h^2+1 & 0 \end{array}\right]}_{\text{Gaussian elimination}}$$

How many solutions?

## *h as Coefficients of Variables*

$x + hy = 0$
$hx + y = 0$
Use Gaussian elimination $-h$ equation 1 + equation 2
We'll use an augmented matrix instead of writing the full equations each time.

Use $x$ term in eq1 to eliminate terms below it via $r_h' = cr_1 + r_k$

$$\left[\begin{array}{ccc} 1 & h & 0 \\ h & 1 & 0 \end{array}\right] \xrightarrow{r_2'=-hr_1+r_2} \left[\begin{array}{ccc} 1 & h & 0 \\ ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc} 1 & h & 0 \\ 0 & -h^2+1 & 0 \end{array}\right]$$

Gaussian elimination

How many solutions? in Maple?

# *h as Coefficients of Variables*

$x + hy = 0$
$hx + y = 0$
Use Gaussian elimination $-h$ equation 1 + equation 2
We'll use an augmented matrix instead of writing the full equations each time.

Use $x$ term in eq1 to eliminate terms below it via $r_h' = cr_1 + r_k$

$$\left[\begin{array}{ccc} 1 & h & 0 \\ h & 1 & 0 \end{array}\right] \xrightarrow{r_2'=-hr_1+r_2} \left[\begin{array}{ccc} 1 & h & 0 \\ ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc} 1 & h & 0 \\ 0 & -h^2+1 & 0 \end{array}\right]$$

Gaussian elimination

How many solutions? in Maple?

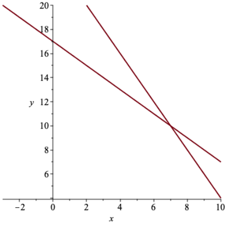
```
heqs:=Matrix([[1,h,0],[h,1,0]]);
GaussianElimination(heqs);
ReducedRowEchelonForm(heqs);
```

# Systems of Equations with 2 Unknowns

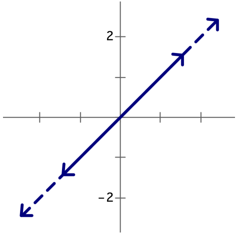
What are the possible solutions of a linear system with two equations and two unknowns? Can you provide examples? More generally, why do these represent all the possibilities?

# Systems of Equations with 2 Unknowns

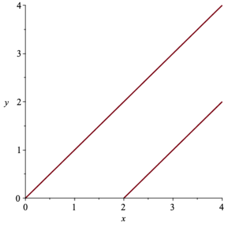
What are the possible solutions of a linear system with two equations and two unknowns? Can you provide examples? More generally, why do these represent all the possibilities?




$$x + y = 17$$
$$4x + 2y = 48$$




$$x - y = 0$$
$$-x + y = 0$$



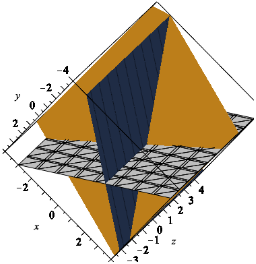

$$x - y = 0$$
$$x - y = 2$$

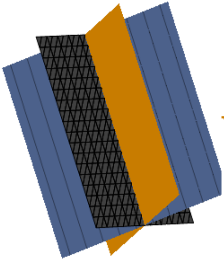
# *Systems of Equations with 3 Unknowns*

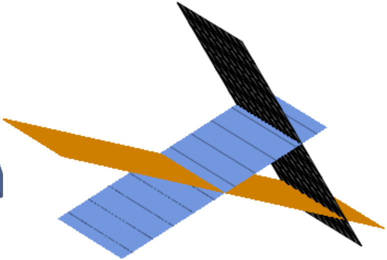
If possible, draw a picture of a linear system with two equations and three unknowns that has a unique solution. If this is not possible, explain why not.

# *Systems of Equations with 3 Unknowns*

If possible, draw a picture of a linear system with two equations and three unknowns that has a unique solution. If this is not possible, explain why not.
How many solutions does each have? Where in the room do we see them?



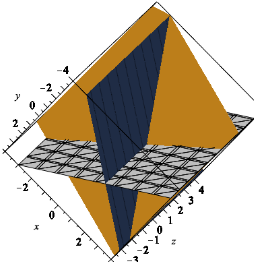





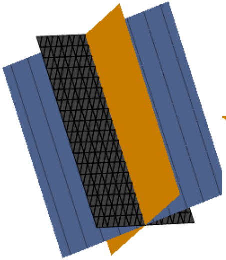
# *Systems of Equations with 3 Unknowns*

If possible, draw a picture of a linear system with two equations and three unknowns that has a unique solution. If this is not possible, explain why not.

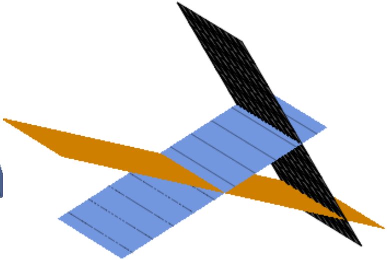
How many solutions does each have? Where in the room do we see them?




1 solution
corner of room



infinite solutions
book spine



0 solutions
hands + table

$$\begin{bmatrix} 3x & & -z & = 3 \\ 2x & -y & +z & = 8 \\ x & 2y & +3z & = 9 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} x & 2y & +3z & = 9 \\ 2x & -y & +z & = 8 \\ 3x & & -z & = 3 \end{bmatrix}$$

$$\left[\begin{array}{llll} 3x & & -z & = 3 \\ 2x & -y & +z & = 8 \\ x & 2y & +3z & = 9 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{llll} x & 2y & +3z & = 9 \\ 2x & -y & +z & = 8 \\ 3x & & -z & = 3 \end{array}\right]$$

$$\xrightarrow{r_2' = -2r_1 + r_2} \left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 3x & & -z & = 3 \end{array}\right]$$

$$\left[\begin{array}{llll} 3x & & -z & = 3 \\ 2x & -y & +z & = 8 \\ x & 2y & +3z & = 9 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{llll} x & 2y & +3z & = 9 \\ 2x & -y & +z & = 8 \\ 3x & & -z & = 3 \end{array}\right]$$

$$\xrightarrow{r_2' = -2r_1 + r_2} \left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 3x & & -z & = 3 \end{array}\right]$$

$$\xrightarrow{r_3' = -3r_1 + r_3} \left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 0 & -6y & -10z & = -24 \end{array}\right]$$

$$\begin{bmatrix} 3x & & -z & = 3 \\ 2x & -y & +z & = 8 \\ x & 2y & +3z & = 9 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} x & 2y & +3z & = 9 \\ 2x & -y & +z & = 8 \\ 3x & & -z & = 3 \end{bmatrix}$$

$$\xrightarrow{r_2' = -2r_1 + r_2} \begin{bmatrix} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 3x & & -z & = 3 \end{bmatrix}$$

$$\xrightarrow{r_3' = -3r_1 + r_3} \begin{bmatrix} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 0 & -6y & -10z & = -24 \end{bmatrix}$$

$$\xrightarrow{r_2' = -\frac{1}{5} r_2} \begin{bmatrix} x & +2y & +3z & = 9 \\ 0 & y & +z & = 2 \\ 0 & -6y & -10z & = -24 \end{bmatrix}$$

$$\begin{bmatrix} 3x & & -z & =3 \\ 2x & -y & +z & =8 \\ x & 2y & +3z & =9 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} x & 2y & +3z & =9 \\ 2x & -y & +z & =8 \\ 3x & & -z & =3 \end{bmatrix}$$

$$\xrightarrow{r_2' = -2r_1 + r_2} \begin{bmatrix} x & +2y & +3z & =9 \\ 0 & -5y & -5z & =-10 \\ 3x & & -z & =3 \end{bmatrix}$$

$$\xrightarrow{r_3' = -3r_1 + r_3} \begin{bmatrix} x & +2y & +3z & =9 \\ 0 & -5y & -5z & =-10 \\ 0 & -6y & -10z & =-24 \end{bmatrix}$$

$$\xrightarrow{r_2' = -\frac{1}{5} r_2} \begin{bmatrix} x & +2y & +3z & =9 \\ 0 & y & +z & =2 \\ 0 & -6y & -10z & =-24 \end{bmatrix}$$

$$\xrightarrow{r_3' = 6r_2 + r_3} \begin{bmatrix} x & +2y & +3z & =9 \\ 0 & y & +z & =2 \\ 0 & 0 & -4z & =-12 \end{bmatrix}$$

The original system has the same solution set as the system

$$\left[\begin{array}{llll} x & +2y & +3z & =9 \\ 0 & y & +z & =2 \\ 0 & 0 & -4z & =-12 \end{array}\right]$$

We can see that $z = 3$. Substituting this into equation 2 gives

The original system has the same solution set as the system

$$\left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & y & +z & = 2 \\ 0 & 0 & -4z & = -12 \end{array}\right]$$

We can see that $z = 3$. Substituting this into equation 2 gives

$$\begin{aligned} y + 3 &= 2 \\ y &= 2 - 3 \\ y &= -1 \end{aligned}$$

So $z = 3$ and $y = -1$. Substituting this into equation 1 gives

The original system has the same solution set as the system

$$\left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & y & +z & = 2 \\ 0 & 0 & -4z & = -12 \end{array}\right]$$

We can see that $z = 3$. Substituting this into equation 2 gives

$$\begin{aligned} y + 3 &= 2 \\ y &= 2 - 3 \\ y &= -1 \end{aligned}$$

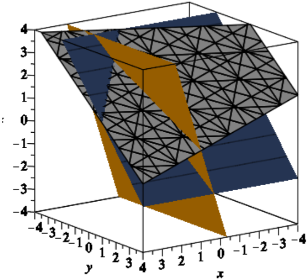
So $z = 3$ and $y = -1$. Substituting this into equation 1 gives

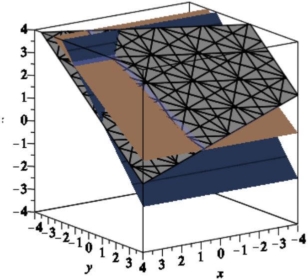
$$\begin{aligned} x + 2(-1) + 3(3) &= 9 \\ x - 2 + 9 &= 9 \\ x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Therefore the unique solution to our linear system is $(x, y, z) = (2, -1, 3)$.

# *Thinking about the Geometry*

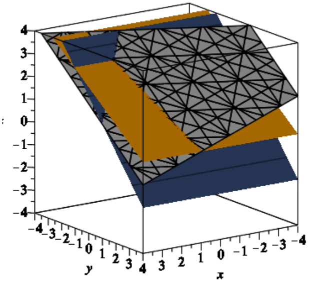
$$\left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 3x & & -z & = 3 \end{array}\right] \qquad \left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 0 & -6y & -10z & = -24 \end{array}\right]$$

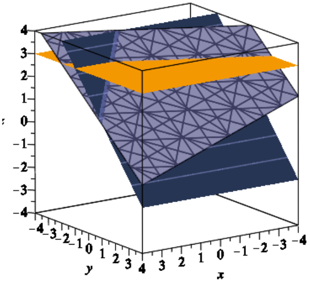


# Thinking about the Geometry

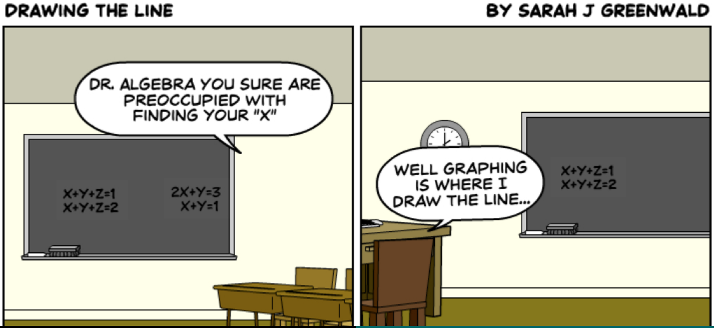
$$\left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & y & +z & = 2 \\ 0 & -6y & -10z & = -24 \end{array}\right] \quad \left[\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & y & +z & = 2 \\ 0 & 0 & -4z & = -12 \end{array}\right]$$



# *Systems of Equations with 3 Unknowns in Maple*

implicitplot3d(3*x-z=3, 2*x-y+z=8, x+2*y+3*z=9,x=-4..4, y = -4..4, z=-4..4);
prob3planes:=Matrix([[3,0,-1,3],[2,-1,1,8],[1,2,3,9]]);
GaussianElimination(prob3planes);
ReducedRowEchelonForm(prob3planes);



# *Solutions?*

A system of equations has an augmented matrix row-equivalent to

$$\begin{matrix} x & y & z & = \end{matrix}$$
$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 7 & -2 \\ 0 & -1 & -7 & 2 \end{bmatrix}$$

How many solutions does the system have?

# *Solutions?*

A system of equations has an augmented matrix row-equivalent to

$$\begin{array}{cccc} x & y & z & = \\ \end{array}$$
$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 7 & -2 \\ 0 & -1 & -7 & 2 \end{bmatrix}$$

How many solutions does the system have?

If we changed the 2 in row 3 column 4 of the given augmented matrix to a 3, then how many solutions does the system have?